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if $x = 1/e = 0.368$ approx. Therefore, the second differences in a table of values of this function must be practically zero for values of x near 0.368.[1]

Also solved by H. L. OLSON, H. S. UHLER, A. H. WILSON, and the Proposer.

2842 [1920, 274]. Proposed by H. S. UHLER, Yale University.

Express explicitly the following sextic in x as the product of a quadratic and a biquadratic:

$$3x^6 - 6k_1x^5 + (7k_1^2 - 9k_2^2)x^4 - 2(2k_1^3 - 4k_1k_2^2 - 3k_3^3)x^3 + [(k_1^2 - k_2^2)^2 - 9k_1k_3^3]x^2$$
$$- (k_1^2 - 2k_2^2)(k_1k_2^2 - 9k_3^3)x + (k_1^2 - 3k_2^2)(k_2^4 - 3k_1k_3^3).$$

I. SOLUTION BY W. D. CAIRNS, Oberlin College.

If $k_1 = k_3 = 0$, the expression can be factored thus:

$$(3x^4 + k_2^4)(x^2 - 3k_2^2).$$

If $k_1 = k_2 = 0$, it can be factored thus:

$$3x^3(x^3 + 2k_3^3) \text{ or } (3x^4 + 6k_3^3x)x^2.$$

If $k_2 = k_3 = 0$, it can be factored thus:

$$x^2(3x^2 - 3k_1x + k_1^2)(x^2 - k_1x + k_1^2).$$

These will be consistent only if one factor contains, at least, the terms $3x^4 - 3k_1x^3 + k_1^2x^2 + 6k_3^3x + k_2^4$, and the second $x^2 - k_1x + k_1^2 - 3k_2^2$. The last term of the given expression shows that the first factor must contain also the term $-3k_1k_3^3$, and multiplication of these two tentative factors shows that the term $-k_1k_2^2x$ completes the first factor. Thus we have as the two factors: $3x^4 - 3k_1x^3 + k_1^2x^2 - (k_1k_2^2 - 6k_3^3)x + (k_2^4 - 3k_1k_3^3)$ and $x^2 - k_1x + (k_1^2 - 3k_2^2)$.

II. REMARKS BY H. P. MANNING, Providence, R. I.

The polynomial can be written $u_3k_3^3 + v_6$, where u_3 and v_6 are polynomials of degrees 3 and 6 in x, k_1 and k_2. If there are rational factors k_3^3 must go entirely with one factor, since u_3 and v_6 are not the cubes of rational expressions. The factor which does not contain k_3 will be the highest common factor of u_3 and v_6. We can also factor u_3, noting that it contains only the second power of k_2.

2845 [1920, 326]. Proposed by E. L. POST, Princeton University.

Prove that if y_x is a solution of the functional equation

$$y_x = \frac{y^2_{x+1}}{x} + y_{x+1}$$

for positive integral values of x with $y_x > 0$, then

$$\lim_{x \to \infty} y_x \log x = 1.$$

SOLUTION BY THE PROPOSER, AND OTTO DUNKEL, Washington University.

If x and y_x are both positive, the equation

$$y_x = \frac{y^2_{x+1}}{x} + y_{x+1} \tag{1}$$

has one positive root,

$$y_{x+1} = \frac{\sqrt{x^2 + 4xy_x} - x}{2}.$$

From (1), we clearly have then that

$$y_x > y_{x+1} > 0. \tag{2}$$

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[1] Since the second difference $y(a + 2h) - 2y(a + h) + y(a) = h^2y''(\theta)$, where $a < \theta < a + 2h$.—EDITORS.

The equation may be written

$$\frac{1}{y_{x+1}} - \frac{1}{y_x} = \frac{y_{x+1}}{xy_x}.$$

Hence

$$\frac{1}{y_x} - \frac{1}{y_1} = \sum_1^{x-1} \frac{1}{x} \cdot \frac{y_{x+1}}{y_x} = \sum_1^{x-1} \frac{1}{x}\left[1 - \frac{y_x - y_{x+1}}{y_x}\right] = \sum_1^{x-1} \frac{1}{x} - \sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2{}_{x+1}}{y_x}.$$

Now

$$\sum_1^{x-1} \frac{1}{x} = \log x + r,$$

where r approaches Euler's constant C when x becomes infinite (Goursat-Hedrick, vol. 1, p. 103, Ex. 1); so that, if we substitute and divide by $\log x$, we have

$$\frac{1}{y_x \log x} - 1 = \frac{\dfrac{1}{y_1} + r - \sum\limits_1^{x-1} \dfrac{1}{x^2} \cdot \dfrac{y^2{}_{x+1}}{y_x}}{\log x}.$$

But from (2) $y_{x+1} < y_x < y_1$, $y^2{}_{x+1} < y_1 y_x$. Therefore

$$\sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2{}_{x+1}}{y_x} < y_1 \sum_1^{x-1} \frac{1}{x^2} < \frac{\pi^2 y_1}{6}.$$

The numerator of the fraction remains numerically less than a fixed number when x becomes infinite and we have

$$\lim y_x \log x = 1.$$

It is to be noticed that the above method can be immediately extended to

$$y_x = f(x) y^2{}_{x+1} + y_{x+1},$$

for certain functions $f(x)$, where $\sum\limits_1^{\infty} [f(x)]^2$ is a convergent series.

NOTE. From (1) we obtain

$$\frac{y_x}{y_{x+1}} - 1 = \frac{y_{x+1}}{x}, \quad \text{and} \quad x\left(\frac{1}{y_{x+1}} - \frac{1}{y_x}\right) = \frac{y_{x+1}}{y_x}. \tag{3}$$

When x becomes infinite, the first equation in (3) together with (2) shows that y_x/y_{x+1} approaches unity. Hence the second equation gives

$$\lim_{x \to \infty} x\left(\frac{1}{y_{x+1}} - \frac{1}{y_x}\right) = 1.$$

It is easily seen that the limit is the same if we replace x by $x + 1$ and when this is done it follows that

$$\lim_{x \to \infty} \frac{1}{\log (x + 1) y_{x+1}} = 1,$$

by use of the theorem on page 108, § 162, E. Cesàro, *Elementares Lehrbuch der algebraischen Analysis* . . ., Leipzig, 1904. The desired result easily follows from the above.

2846 [1920, 326].

Find the entire volume within the surface $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$. (W. A. Granville, *Elements of Differential and Integral Calculus*, revised ed., 1911, p. 420.)

This equation, rationalized, is the equation of Steiner's quartic surface, every tangent plane to which cuts it in two conics. (Cf. Salmon-Rogers, *Analytic Geometry of Three Dimensions*, 5th ed., vol. 2, 1915, pp. 171, 201, 207, 213f. Also C. M. Jessop, *Quartic Surfaces* 1916, chapter 7.)

I. SOLUTION BY L. A. EASTBURN, North Arizona Normal School, Flagstaff, Ariz.

The required volume inclosed by the surface is

$$v = \int_0^a \int_0^{(a^{1/2}-x^{1/2})^2} \int_0^{(a^{1/2}-x^{1/2}-y^{1/2})^2} dz\,dy\,dx,$$